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Two non-probabilistic set-theoretical models for dynamic response and buckling failure measures of bars with unknown-but-bounded initial imperfections

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Abstract

This paper is concerned with the problem of comparison of two non-probabilistic set-theoretical models for dynamic response and buckling failure measures of bars with unknown-but-bounded initial imperfections. Two kinds of non-probabilistic set-theoretical models are convex models and interval analysis models. In convex models and interval analysis models, the uncertain quantities are considered to be unknown except that they belong to given sets in an appropriate vector space. In this case, all information about the dynamic response and buckling failure measures of bars is provided by the set of dynamic responses and buckling failure measures consistent with the constraints on the uncertain quantities. The dynamic response estimate is actually a set in appropriate response space rather than a single vector. The set estimate is the smallest calculable set which contains the uncertain dynamic response, but it is usually impractical to calculate this set. Two set estimate methods are developed which can calculate the time varying box or hyperrectangle, i.e. interval vector in the response space that contains the true dynamic response. Comparison between convex models and interval analysis models in mathematical proofs and numerical calculations shows that, under the condition of the outer enclosed ellipsoid from a hyperrectangle or an interval vector, the set dynamic response predicted by interval analysis models is smaller than that yielded by convex models; under the condition of the outer enclosed hyperrectangle or an interval vector from an ellipsoid, the dynamic response set calculated by convex models is smaller than that obtained by interval analysis models.

Keywords: Dynamic response; Buckling failure; Non-probabilistic set-theoretical models; Unknown-but-bounded initial imperfections

1. Introduction

It is well known that initial imperfections in structures, inevitably present due to the very manufacturing process, have a significant effect on the buckling problem of structures, in particular for thin bars. Indeed it

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is unlikely that two bars, even produced by the same manufacturing procedure, will possess the same deviations from the nominally straight state. Up to now, there is a considerable body of literature (Koning and Taub, 1934; Malyshev, 1966; Lindberg, 1965, 1991; Elishakoff, 1978a,b; Ben-Haim and Elishakoff, 1989, 1990) dealing with the dynamic response and buckling failure measures of bars. In a number of studies (Koning and Taub, 1934; Malyshev, 1966) a deterministic model of initial imperfections is developed, and the initial imperfections are expressed in the form of a half sine wave, which is the buckling mode of the corresponding static problem of bars. Some results showed that, if the external load is greater than the classical buckling load, the bar deflection grows exponentially in time and remains freely oscillating after unloading, and that the amplitude of the oscillation is equal to the maximum deflection. If the external load is less than the Euler buckling load then the maximum deflection can occur after unloading. However, the initial imperfections are not necessarily proportional to the classical buckling mode, but rather are functions of the involving uncertainties. This kind of uncertainty was considered to have randomness and was studied by Lindberg (1965) using probabilistic theory. In the probabilistic models, the initial imperfection function was expanded in a Fourier series in terms of the classical buckling modes. The Fourier coefficients were assumed to be normally distributed random variables with zero mean value and with variance, which was assumed to be band-limited white-noise, proportional to the power spectral density of the initial imperfection. Elishakoff (1978a,b) treated the same problem from the structural reliability point of view. Closed form results were obtained for the case when the initial imperfection was co-configured with the buckling mode, whereas for the general case the solution was reported through the Monte Carlo method. Obviously, if sufficient probabilistic information is available, then probabilistic approach enable one to evaluate the dynamic response and buckling failure measures of bars, which is of foremost importance in design and analysis. However, in many cases, the probabilistic information on initial imperfections, which is needed to determine the dynamic response and buckling failure measures, is unfortunately unavailable. Under the circumstance, an alternative, non-probabilistic, unknown-but-bounded initial imperfection model was introduced by Ben-Haim and Elishakoff (1989, 1990) for the study of dynamic response and failure of elastic bars under dynamic axial loads. Imperfections were taken as a series of N terms of the natural vibration and buckling mode shapes of the bar, with coefficients specified to be bounded by an ellipsoid in N-dimensional Euclidean space. Buckling failure was defined by the deflection of the bar at a particular position and time. For each measure of failure, a formula was derived for the maximum possible dynamic response for any imperfection vector within or on the ellipsoid. Independent of these developments, another approach has been to specify the imperfections by a bounding box or hyperrectangle. In this model, all initial imperfections are assumed to be bounded from above and below. Based on interval mathematics, in the studies by Qiu et al. (1996, 2001a,b), Rao and Berke (1997), interval analysis methods of uncertainty were developed for modeling uncertain parameters of structures. In recent work of non-probabilistic convex models (Ben-Haim and Elishakoff, 1989, 1990; Qiu et al., 2001b; Pantelides and Ganerli, 2001) and interval analysis methods (Qiu et al., 1995; Qiu and Elishakoff, 1998), bounds on the magnitude of uncertain variables are only required, not necessarily knowing the probabilistic distribution densities, following the general methodologies developed in the monographs. It was assumed that the uncertain variables fall into the multidimensional box, instead of conventional optimization studies, where the minimum possible response is sought, here an uncertainty modeling is developed as a optimization problem of finding the least favorable response and the most favorable response under the constrains within the set-theoretical description. Convex models and interval analysis methods have been used for dealing with uncertain problems in a wide range of engineering applications.

In this paper, from mathematical proofs and numerical calculations, the comparison between non-probabilistic convex model and interval analysis model was carried out. Relationship between two models and their respective measures of dynamic response are demonstrated which give better insight into the two models to deal with uncertain problems.

2. Statement of the problem

The differential equation of motion for the bar is given by

$$EI\frac{\partial^4 y}{\partial x^4}+P\frac{\partial^2 y}{\partial x^2}+\rho A\frac{\partial^2 y}{\partial t^2}=-P\frac{\partial^2 \hat{y}}{\partial x^2} \tag{1}$$

where x is the axial coordinate, t is the time, $\hat{y}(x)$ is the initial imperfection, i.e. a small perturbation to the initial shape of the compressed bar, $y(x,t)$ is the additional transverse deflection measured from $\hat{y}(x)$ ($\hat{y}(x)+y(x,t)$ being the total deflection of the bar axis from the straight line between the two ends $x=0$ and $x=l$), E is Young's modulus, I is the moment of inertia, ρ is the mass density, A is the cross-sectional area, P is the applied axial load, EI and ρA are taken as constant.

The compressed bar is simply supported, so that the boundary conditions are

$$y=0,\quad \frac{\partial^2 y}{\partial x^2}=0 \text{ at } x=0 \text{ and } x=l \tag{2}$$

and the initial conditions are

$$y=0,\quad \frac{\partial y}{\partial t}=0 \text{ at } t=0 \tag{3}$$

The initial imperfections are assumed to be an uncertain function of the position x and are bounded.

We now introduce the non-dimensional quantities

$$u(\xi,\tau)=\frac{y(x,t)}{\Delta},\quad \hat{u}(\xi)=\frac{\hat{y}(x)}{\Delta},\quad \xi=\frac{x}{l},\quad \tau=\omega_1 t,\quad \alpha=\frac{P}{P_{\text{cl}}} \tag{4}$$

where

$$\Delta=\sqrt{\frac{I}{A}},\quad P_{\text{cl}}=\frac{\pi^2 EI}{l^2},\quad \omega_1=\left(\frac{\pi}{l}\right)^2\sqrt{\frac{EI}{\rho A}} \tag{5}$$

in which $u(\xi,\tau)$ is the non-dimensional additional displacement, $\hat{u}(\xi)$ is the non-dimensional initial displacement, ξ is the axial coordinate, τ is the time, α is the non-dimensional applied load, P_{cl} is the classical buckling load of a perfect bar, ω_1 is the first eigenfrequency of a perfect bar without an axial load, Δ is the radius of gyration of the bar cross section.

Thus, the differential equation (1), the boundary and initial conditions become, respectively,

$$\frac{\partial^4 u}{\partial \xi^4}+\pi^2\alpha\frac{\partial^2 u}{\partial \xi^2}+\pi^4\frac{\partial^2 u}{\partial \tau^2}=-\pi^2\alpha\frac{\partial^2 \hat{u}}{\partial \xi^2} \tag{6}$$

and

$$u=0,\quad \frac{\partial u}{\partial \tau}=0 \text{ at } \tau=0 \tag{7}$$

and

$$u=0,\quad \frac{\partial^2 u}{\partial \xi^2}=0 \text{ at } \xi=0 \text{ and } \xi=1 \tag{8}$$

For this uncertain boundary-value problem we resort to the normal mode approach, which consists in expanding the given and sought functions in series in terms of the modes of free vibration of a perfect bar without an axial load.

The constant load P is applied at time $\tau = 0$ and kept indefinitely. We expand the initial imperfection profile in a Fourier sine series as

$$\hat{u}(\xi) = \sum_{n=1}^{\infty} A_n \sin n\pi\xi \tag{9}$$

where A_n's are the uncertain variables and bounded.

We expand the additional deflection $u(\xi, \tau)$ in a series in the same manner

$$u(\xi, \tau) = \sum_{n=1}^{\infty} G_n(\tau) \sin n\pi\xi \tag{10}$$

where $G_n(\tau)$ is an uncertain function of τ.

Following Elishakoff (1978a,b), one finds that the Fourier coefficients of the additional deflection profile are

$$G_n(\tau) = A_n \psi_n(\tau) \tag{11}$$

in which ψ_n is the following deterministic function of time:

$$\psi_n(\tau) = \begin{cases} (\cosh(r_n\tau) - 1)/(1 - \beta_n), & \beta_n < 1 \\ \alpha n^2 \tau^2/2, & \beta_n = 1 \\ (\cos(r_n\tau) - 1)/(1 - \beta_n), & \beta_n > 1 \end{cases} \tag{12}$$

where

$$\beta_n = \frac{n^2}{\alpha}, \quad r_n = n\sqrt{|n^2 - \alpha|} \tag{13}$$

Thus, the total normalized deflection function $v(\xi, \tau)$ at position ξ and at time τ is given by

$$v(\xi, \tau) = u(\xi, \tau) + \hat{u}(\xi) = \sum_{n=1}^{\infty} A_n(1 + \psi_n(\tau)) \sin n\pi\xi \tag{14}$$

In the subsequent discussion this series will be truncated at $n = N$. For convenience, let us define an N-dimensional vector as follows:

$$\varphi = (\varphi_i), \quad \varphi_i(\xi, \tau) = (1 + \psi_i(\tau)) \sin n\pi\xi \tag{15}$$

Thus, truncating the total deflection $v(\xi, \tau)$ in Eq. (14) after the Nth term, one finds

$$v(\xi, \tau) = \sum_{n=1}^{N} A_n(1 + \psi_n(\tau)) \sin(n\pi\xi) = A^{\mathrm{T}}\varphi \tag{16}$$

where $A^{\mathrm{T}} = (A_1, A_2, \ldots, A_N)$.

The aim of this paper is to exploit two non-probabilistic set-theoretical models to determine the region or the upper and lower bounds of the parameter characterizing the total normalized deflection of a thin bar under dynamic loading conditions, given uncertain-but-bounded initial geometric imperfection region. By evaluating the region of the dynamic response parameter as a function of the load and of the uncertainties in the initial imperfection of the bar, it is possible to identify the allowable imperfection region for given region of the load.

3. Convex models

A new, non-probabilistic, convex models of uncertainty have been developed (Koning and Taub, 1934), for applied mechanics applications in a quite general context. The models utilize the imperfect, scanty knowledge on uncertain quantities, instead of precise information on the probability contents of random variables. The models use a representation of uncertainty phenomena by convex sets such as ellipsoid. The approach itself is also referred to a convex modeling.

Let $A_0 = (A_{n0})$ be the nominal values of the uncertain-but-bounded initial imperfection Fourier coefficients in Eq. (16), which might be visualized as the mean values or average values of those uncertain-but-bounded initial imperfection Fourier coefficients. Then, the uncertain-but-bounded initial imperfection Fourier coefficients of values different from those nominal values could be denoted as the following vector form

$$A = A_0 + \delta \tag{17a}$$

and the component form

$$A_n = A_{n0} + \delta_n, \quad n = 1, 2, \ldots, N \tag{17b}$$

where $\delta = (\delta_n)$ is the uncertain quantity in the uncertain-but-bounded initial imperfection Fourier coefficients. Thus, the total normalized deflection function $v(\xi, \tau)$ can be rewritten as

$$\begin{aligned} v(\xi, \tau) &= \sum_{n=1}^{N} A_{n0}(1 + \psi_n(\tau)) \sin(n\pi\xi) + \sum_{n=1}^{N} \delta_n(1 + \psi_n(\tau)) \sin(n\pi\xi) \\ &= A_0^{\mathrm{T}}\varphi + \delta^{\mathrm{T}}\varphi = v_0(\xi, \tau) + \delta^{\mathrm{T}}\varphi \end{aligned} \tag{18}$$

where

$$v_0(\xi, \tau) = \sum_{n=1}^{N} A_{n0}(1 + \psi_n(\tau)) \sin(n\pi\xi) = A_0^{\mathrm{T}}\varphi \tag{19}$$

Let us assume that we have only limited information for characterizing the initial imperfection profile. In particular, the only information is that the deviation $\delta = (\delta_n)$ from the initial imperfection Fourier coefficients in Eq. (14) fall within the following ellipsoidal set

$$E(\theta, W) = \{\delta^{\mathrm{T}} = (\delta_1, \delta_2, \ldots, \delta_N) : \delta^{\mathrm{T}} W \delta \leqslant \theta^2\} \tag{20}$$

where W is an $N \times N$-dimensional positive definite real symmetric matrix and is called the weighting matrix, and θ is a positive number and is called the radius of the ellipsoid. The shape and size of the ellipsoid are determined by the weighted matrix W and the radius θ, which are chosen to represent available information concerning the variability of the Fourier coefficients of the initial deflection profile. N is the number of dominant Fourier coefficients.

The bounded convex set (20) may be thought of as a constraint condition. Thus, by solving for the total deflection of the bar $v(\xi, \tau)$ we mean to solve the family of the differential equations in which the initial geometric imperfection is uncertain ranging in inside the certain convex set. That is to say that the total deflection of the bar equation with initial geometric imperfection is a set. Taking this into account, one has to determine a closed convex interval set $[v_{\min}(\xi, \tau), v_{\max}(\xi, \tau)]$ for the total deflection, one of the smallest widths enclosing all possible values of the total normalized deflection $v(\xi, \tau)$. Because $E(\theta, W)$ represents a realistic ensemble of the initial imperfection Fourier coefficients, the lowest total deflection $v_{\min}(\xi, \tau)$ which can be obtained from any of the bars in this ensemble is expressed formally as the minimum of expression (18) on the set $E(\theta, W)$.

The set of extreme points of the ellipsoidal set $E(\theta, W)$ is the ellipsoidal shell

$$S(\theta, W) = \{\delta^{\mathrm{T}} = (\delta_1, \delta_2, \ldots, \delta_N) : \delta^{\mathrm{T}} W \delta = \theta^2\} \tag{21}$$

The problem is formulated as follows: given an imperfection ellipsoid of the initial imperfections, find the initial imperfection vector that extremize the total deflection.

The extremum on the convex set $E(\theta, W)$ of the total deflection of the bar at time τ and position ξ is represented as

$$v_{\mathrm{ext}}(\xi, \tau) = \underset{A \in E(\theta, W)}{\mathrm{extremum}} \{v(\xi, \tau)\} \tag{22}$$

Because $v(\xi, \tau)$ is a linear function of the deviation vector in the Fourier coefficients $\delta = (\delta_n)$, and because $E(\theta, W)$ is a convex set, the extremum of the total deflection $v(\xi, \tau)$ occurs on the set of extreme points of the set $E(\theta, W)$.

That is to say

$$v_{\mathrm{ext}}(\xi, \tau) = \underset{A \in S(\theta, W)}{\mathrm{extremum}} \{v(\xi, \tau)\} \tag{23}$$

We can obtain an explicit expression for the extremum by employing the method of Lagrange multipliers. In the most cases, the weighting matrix of the ellipsoid is often taken as the following $N \times N$-dimensional diagonal matrix form

$$W = \mathrm{dia}\left(\frac{1}{e_n^2}\right) \tag{24}$$

where $e_n > 0$, $n = 1, 2, \ldots, N$. Then, the ellipsoidal equation can be written as

$$\delta^{\mathrm{T}} W \delta = \sum_{n=1}^{N} \frac{\delta_n^2}{e_n^2} \leqslant \theta^2 \tag{25}$$

Thus, by convex models, the least favorable and the most favorable total deflections may be determined as follows:

$$v_{\max}(\xi, \tau) = \sum_{n=1}^{N} A_{n0}(1 + \psi_n(\tau)) \sin(n\pi\xi) + \theta \sqrt{\sum_{n=1}^{N} (e_n(1 + \psi_n(\tau)) \sin(n\pi\xi))^2} \tag{26a}$$

and

$$v_{\min}(\xi, \tau) = \sum_{n=1}^{N} A_{n0}(1 + \psi_n(\tau)) \sin(n\pi\xi) - \theta \sqrt{\sum_{n=1}^{N} (e_n(1 + \psi_n(\tau)) \sin(n\pi\xi))^2} \tag{26b}$$

Under dynamic loading conditions, by Eq. (26) we can calculate the maximum and minimum bounds of the total normalized deflection of a thin bar with given uncertain-but-bounded initial geometric imperfection region using the convex models.

4. Interval analysis method

In the total normalized deflection function $v(\xi, \tau)$, considering a realistic situation in which available information on the dominant N initial imperfection Fourier coefficients A_n, $n = 1, 2, \ldots, N$ in Eq. (16) is not enough to justify an assumption on their probabilistic characteristics, we follow the thought of interval mathematics, and assume that the dominant N initial imperfection Fourier coefficients $A_n, n = 1, 2, \ldots, N$ are within bounding vectors $\overline{A} = (\overline{A}_1, \overline{A}_2, \ldots, \overline{A}_N)^{\mathrm{T}}$ and $\underline{A} = (\underline{A}_1, \underline{A}_2, \ldots, \underline{A}_N)^{\mathrm{T}}$, respectively:

$$\underline{A}_n \leqslant A_n \leqslant \overline{A}_n, \quad n = 1, 2, \ldots, N \tag{27}$$

Clearly, the above inequalities can be expressed as the following interval vector form (Alefeid and Herzberger, 1983; Moore, 1979; Deif, 1991):

$$A^{\mathrm{I}} = [\underline{A}, \overline{A}] = (A_n^{\mathrm{I}}), \quad A_n^{\mathrm{I}} = [\underline{A}_n, \overline{A}_n], \quad n = 1, 2, \ldots, N \tag{28}$$

where $\overline{A} = (\overline{A}_n)$ and $\underline{A} = (\underline{A}_n)$.

From the interval vector equations (28), we may define the nominal value vector or midpoint vector or center vector of the interval dominant N initial imperfection Fourier coefficient vector

$$A^{\mathrm{c}} = \frac{(\overline{A} + \underline{A})}{2} = (A_n^{\mathrm{c}}), \quad A_n^{\mathrm{c}} = \frac{(\overline{A}_n + \underline{A}_n)}{2}, \qquad n = 1, 2, \ldots, N \tag{29}$$

and the deviation amplitude vector or radius vector or uncertainty of the interval dominant N initial imperfection Fourier coefficient vector

$$\Delta A = \frac{(\overline{A} - \underline{A})}{2} = (\Delta A_n), \quad \Delta A_n = \frac{(\overline{A}_n - \underline{A}_n)}{2}, \qquad n = 1, 2, \ldots, N \tag{30}$$

In terms of the interval arithmetic operations, an arbitrary interval number or vector or matrix can be written as the sum of its midpoint and its uncertain interval. Thus, the interval dominant N initial imperfection Fourier coefficient vector $A^{\mathrm{I}} = [\underline{A}, \overline{A}]$ is decomposed into the sum of the nominal value A^{c} and the deviation amplitude ΔA, i.e.,

$$A^{\mathrm{I}} = [\underline{A}, \overline{A}] = [A^{\mathrm{c}} - \Delta A, A^{\mathrm{c}} + \Delta A] = A^{\mathrm{c}} + [-\Delta A, \Delta A] = A^{\mathrm{c}} + \Delta A^{\mathrm{I}} \tag{31}$$

where $\overline{A} = A^{\mathrm{c}} + \Delta A$, $\underline{A} = A^{\mathrm{c}} - \Delta A$, and

$$\Delta A^{\mathrm{I}} = [-\Delta A, \Delta A] = \Delta A[-1, 1] = \Delta A e_{\Delta} \tag{32}$$

where $e_{\Delta} = [-1, 1]$.

In Eq. (16), the total deflection $v(\xi, \tau)$ is thought of as a function of the dominant N initial imperfection Fourier coefficient A_n, $n = 1, 2, \ldots, N$, by means of the interval extension of interval mathematics, from Eq. (16), we have that

$$v^{\mathrm{I}}(\xi, \tau) = [\underline{v}(\xi, \tau), \overline{v}(\xi, \tau)] = (A^{\mathrm{I}})^{\mathrm{T}} \varphi \tag{33}$$

Substitution of Eq. (31) into the above expression yielding

$$v^{\mathrm{I}}(\xi, \tau) = (A^{\mathrm{c}} + [-\Delta A, \Delta A])^{\mathrm{T}} \varphi \tag{34}$$

Applying the interval operation, Eq. (34) becomes

$$v^{\mathrm{I}}(\xi, \tau) = [\underline{v}(\xi, \tau), \overline{v}(\xi, \tau)] = A^{\mathrm{c}} \varphi + [-\Delta A, \Delta A])^{\mathrm{T}} |\varphi| = A^{\mathrm{c}} \varphi + [-\Delta A^{\mathrm{T}} |\varphi|, \Delta A^{\mathrm{T}} |\varphi|] \tag{35}$$

According to the necessary and sufficient conditions of equality of two interval vectors, we obtain

$$\overline{v}(\xi, \tau) = A^{\mathrm{c}} \varphi + \Delta A^{\mathrm{T}} |\varphi| \tag{36a}$$

and

$$\underline{v}(\xi, \tau) = A^{\mathrm{c}} \varphi - \Delta A^{\mathrm{T}} |\varphi| \tag{36b}$$

In terms of the definition of ΔA and φ, we can deduce

$$\overline{v}(\xi,\tau) = v_c(\xi,\tau) + \sum_{n=1}^{N} \Delta A_n |(1+\psi_n(\tau))\sin(n\pi\xi)| \tag{37a}$$

and

$$\underline{v}(\xi,\tau) = v_c(\xi,\tau) - \sum_{n=1}^{N} \Delta A_n |(1+\psi_n(\tau))\sin(n\pi\xi)| \tag{37b}$$

where

$$v_c(\xi,\tau) = A^c\varphi = \sum_{n=1}^{N} A_n^c (1+\psi_n(\tau))\sin(n\pi\xi) \tag{38}$$

With the expression of A^c and φ, we also conclude

$$\overline{v}(\xi,\tau) = \sum_{n=1}^{N} A_n^c (1+\psi_n(\tau))\sin(n\pi\xi) + \sum_{n=1}^{N} \Delta A_n |(1+\psi_n(\tau))\sin(n\pi\xi)| \tag{39a}$$

and

$$\underline{v}(\xi,\tau) = \sum_{n=1}^{N} A_n^c (1+\psi_n(\tau))\sin(n\pi\xi) - \sum_{n=1}^{N} \Delta A_n |(1+\psi_n(\tau))\sin(n\pi\xi)| \tag{39b}$$

By Eqs. (39) we can determine the interval region of the total normalized deflection of a thin bar under dynamic loading conditions, given uncertain-but-bounded initial geometric imperfection region using the interval analysis method.

5. Comparison of two set-theoretical methods based on determination of the outer enclosed hyperrectangle or interval vector from an ellipsoid

In this section, we will process the comparison problem of the interval analysis method and convex models based on determining the outer enclosed hyperrectangle or interval vector from an ellipsoid.

5.1. Determination of the interval vector from an ellipsoid

Suppose that from the experimental data we can deduce that the uncertain parameters are varying in the ellipsoidal set (20). In order to determine the interval vector from the ellipsoid (20), we suppose that the interval vector (28) that encloses the ellipsoid (20), in two-dimensional space, see Fig. 1.

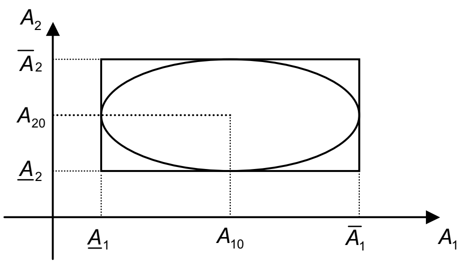

Fig. 1. The rectangle enclosing an ellipse.

According to the symmetry, we can deduce that

$$A^c = A_0 \text{ or } A_n^c = A_{n0}, \quad n = 1, 2, \ldots, N \tag{40}$$

and

$$\Delta A = \theta e \text{ or } \Delta A_n = \theta e_n, \quad n = 1, 2, \ldots, N \tag{41}$$

Eqs. (40) and (41) imply that the midpoint of the interval vector is equal to the mean value of the ellipsoid, the uncertainties of the interval vector are the multiplication of the radius and the semi-axes of the ellipsoid.

5.2. Comparison of two set-theoretical methods

The maximum and minimum values, $v_{\max}(\xi,\tau)$ and $v_{\min}(\xi,\tau)$ of the total deflection of a thin bar with given uncertain initial geometric imperfection, which are obtained by convex models, can be composed to a convex set or an interval

$$v^I(\xi,\tau) = [v_{\min}(\xi,\tau), v_{\max}(\xi,\tau)] \tag{42}$$

From the convex models point of view, the convex set $v^I(\xi,\tau)$ is the smallest set containing all true the total deflection of a thin bar with uncertain initial geometric imperfections.

By interval analysis method, the upper bound $\overline{v}(\xi,\tau)$, and lower bound $\underline{v}(\xi,\tau)$, on the total deflection of a thin bar with uncertain initial geometric imperfections can also constitute an interval number. According to the interval mathematical theory (Ben-Haim and Elishakoff, 1989, 1990), the interval number

$$u^I(\xi,\tau) = [\underline{v}(\xi,\tau), \overline{v}(\xi,\tau)] \tag{43}$$

is also the smallest set containing all total deflections of a thin bar with uncertain initial geometric imperfections.

The following part of this section, we will compare $u^I(\xi,\tau)$ and $v^I(\xi,\tau)$, and show which is the true smaller set in $u^I(\xi,\tau)$ and $v^I(\xi,\tau)$.

Now we suppose that the interval vector $A^I = [\underline{A}, \overline{A}] = (A_n^I)$, $A_n^I = [\underline{A}_n, \overline{A}_n]$, $n = 1, 2, \ldots, N$ of the uncertain initial geometric imperfection is obtained based on the ellipsoid (20) of the uncertain initial geometric imperfection. Thus, by means of Eq. (40), from Eqs. (19) and (38), we can obtain the following equality:

$$v_0(\xi,\tau) = v_c(\xi,\tau) \tag{44}$$

For the uncertainties $\Delta A_n = (\overline{A}_n - \underline{A}_n)/2$, $n = 1, 2, \ldots, N$, of the interval vector A^I and the radius θ and the semi-axes e_n, $n = 1, 2, \ldots, N$ of the ellipsoid of the uncertain-but-bounded initial imperfections vector, from Eqs. (27) and (41) we have that

$$\theta\sqrt{\sum_{n=1}^{N}(e_n(1+\psi_n(\tau))\sin(n\pi\xi))^2} = \theta\sqrt{\sum_{n=1}^{N}(e_n(1+\psi_n(\tau))\sin(n\pi\xi))^2} = \sqrt{\sum_{n=1}^{N}(\Delta A_n(1+\psi_n(\tau))\sin(n\pi\xi))^2} \tag{45}$$

Since

$$\sum_{\substack{m,n=1\\ m\neq n}}^{N} |(\Delta A_m(1+\psi_m(\tau))\sin(m\pi\xi))||(\Delta A_n(1+\psi_n(\tau))\sin(n\pi\xi))|\Delta A_m\Delta A_n$$
$$= \sum_{\substack{m,n=1\\ m\neq n}}^{N} |((1+\psi_m(\tau))\sin(m\pi\xi))||((1+\psi_n(\tau))\sin(n\pi\xi))|\Delta A_m\Delta A_n \geqslant 0 \tag{46}$$

and if $N \geqslant 1$, then, from the following equality:

$$\Delta A^{\mathrm{T}}|\phi| = \sum_{n=1}^{N}|(\Delta A_n(1+\psi_n(\tau))\sin(n\pi\xi))| = \sum_{n=1}^{N}|((1+\psi_n(\tau))\sin(n\pi\xi))|\Delta A_n$$
$$= \sqrt{\left(\sum_{n=1}^{N}|((1+\psi_n(\tau))\sin(n\pi\xi))|\Delta A_n\right)^2}$$
$$= \sqrt{\sum_{n=1}^{N}(|((1+\psi_n(\tau))\sin(n\pi\xi))|\Delta A_n)^2 + \sum_{\substack{m,n=1\\ m\neq n}}^{N}|((1+\psi_m(\tau))\sin(m\pi\xi))||((1+\psi_n(\tau))\sin(n\pi\xi))|\Delta A_m\Delta A_n} \tag{47}$$

We obtain the following inequality:

$$\Delta A^{\mathrm{T}}|\phi| = \sum_{n=1}^{N}|(\Delta A_n(1+\psi_n(\tau))\sin(n\pi\xi))| = \sum_{n=1}^{N}|((1+\psi_n(\tau))\sin(n\pi\xi))|\Delta A_n$$
$$= \sqrt{\left(\sum_{n=1}^{N}|((1+\psi_n(\tau))\sin(n\pi\xi))|\Delta A_n\right)^2} \geqslant \sqrt{\sum_{n=1}^{N}(|((1+\psi_n(\tau))\sin(n\pi\xi))|\Delta A_n)^2}$$
$$= \sqrt{\sum_{n=1}^{N}(\theta e_n((1+\psi_n(\tau))\sin(n\pi\xi)))^2} \tag{48}$$

Because of Eqs. (43) and (48), from Eqs. (30) and (39) we can deduce

$$\underline{v} \leqslant \underline{u}_{\min} \leqslant u_{\max} \leqslant \overline{v} \quad \text{or} \quad u^{\mathrm{I}} \subseteq v^{\mathrm{I}} \tag{49}$$

The expression (53) show that under the condition of the interval vector determined from an ellipsoid of the initial geometric imperfection vector, the total deflection set obtained by convex models is smaller that by interval analysis method for a thin bar with uncertain-but-bounded initial geometric imperfections. Namely the lower bound of the total normalized deflection within interval analysis method is smaller than one predicted by convex models, and the upper bound of the total deflection furnished by the interval analysis method is larger than those yielded by convex models.

6. Comparison of two set-theoretical methods based on determination of the out enclosed ellipsoid from an hyperrectangle or interval vector

In this section, we will handle the comparison problem of the interval analysis method and convex models based on determining the outer enclosed ellipsoid from a hyperrectangle or an interval vector.

6.1. Determination of the ellipsoid form an interval vector

Before any prediction can be made by convex model method and interval analysis method on the buckling critical load of the composite structures with uncertain parameters, the values of the ellipsoidal radius α, the ellipsoidal semi-axes $e = (e_n)_N$, and the upper bound $\overline{A} = (\overline{A}_n)_N$ and lower bound $\underline{A} = (\underline{A}_n)_N$ of the interval vector should be determined in advance. In fact, these values which describe uncertainties in structural parameters are dependent on the manufacturing process by which composite structures have been fabricated. It is understandable that the more advanced the manufacturing process and the better the workmanship, the smaller these uncertainties in structural parameters will be in value. Some measurements of the structural parameters for the composite materials were described by Elishakoff (1978a,b), which shows clearly a scatter or an uncertainty of the measured data for the structural parameters. In particular, for the transverse and longitudinal Poisson's ratio, the experimental values have a large uncertainty. Generally speaking, if sufficient amount of experimental data are available, the average value of these data could be used as the nominal value $A_0 = (A_{n0})_N$ for the corresponding structural parameters, whereas the absolute values of the maximum uncertainties in structural parameters could be chosen as the proper deviations from the average values of the corresponding measured data.

Assume that from the experimental data we can know that the uncertain-but-bounded initial imperfection Fourier coefficients on Eq. (16) are varying in the inequalities (27) respectively. Let $A_0 = (A_{n0})_N$ be the nominal values of the initial imperfection Fourier coefficients. Then, the uncertain-but-bounded initial imperfection Fourier coefficients of values different from those nominal values could be denoted as the following vector form:

$$A = A_0 + \delta, \quad |\delta| \leqslant \Delta A \tag{50a}$$

and the component form

$$A_n = A_{n0} + \delta_n, \quad |\delta_n| \leqslant \Delta A_n, \quad n = 1, 2, \ldots, N \tag{50b}$$

where the second of the above equation describes a box or hyperrectangle. In two-dimensional case, see Fig. 2. In order to give the relation expression of an interval vector and an ellipsoid, we need to enclose this box by an ellipsoid, i.e.

$$\sum_{n=1}^{N} \frac{\delta_n^2}{(\theta e_n)^2} = \sum_{n=1}^{N} \frac{(A_n - A_{n0})^2}{(\theta e_n)^2} \leqslant 1 \tag{51}$$

Obviously, according to the symmetry the interval and the ellipsoid, we can deduce that

$$A_0 = A^{c} \text{ or } A_{n0} = A_n^{c}, \quad n = 1, 2, \ldots, N \tag{52}$$

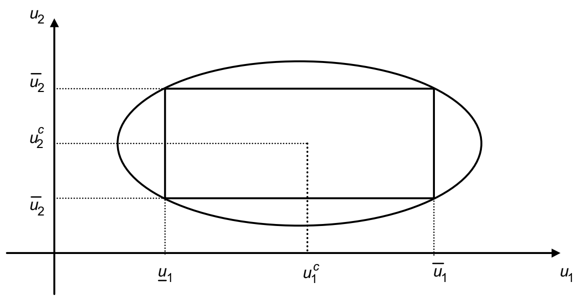

Fig. 2. The ellipse enclosing a rectangle.

Thus, the question arises as to how to determine the semi-axes $e = (e_n)_N$ and the radius θ of this ellipsoid from given the upper bound $\overline{A} = (\overline{A}_n)_N$ and lower bound $A = (A - n)_N$ of the interval vector. Naturally, such an ellipsoid should have a minimum volume of the above ellipsoid is given by

$$V = C\prod_{n=1}^{N}(\theta e_n) \tag{53}$$

where C is a constant.

Since the corner points $A_{n0} \pm \Delta A_n$ of the box (50b) should be on the surface of the ellipsoid, we have that

$$\frac{\Delta A_1^2}{(\theta e_1)^2} + \frac{\Delta A_2^2}{(\theta e_2)^2} + \ldots + \frac{\Delta A_N^2}{(\theta e_N)^2} = 1 \tag{54}$$

Thus, the problem of determining the values θ and $e = (e_n)_N$ from the given values $\overline{A} = (\overline{A}_n)_N$ and $\underline{A} = (\underline{A}_n)_N$ become the minimum value problem of the volume V of the ellipsoid subject to the constraint condition (54). To do this, we use the Lagrange multiplier technique. The Lagrangean function L reads

$$L = C\prod_{n=1}^{N}(\theta e_n) + \lambda\left(\frac{\Delta A_1^2}{(\theta e_1)^2} + \frac{\Delta A_2^2}{(\theta e_2)^2} + \ldots + \frac{\Delta A_N^2}{(\theta e_N)^2} - 1\right) \tag{55}$$

Requirements

$$\frac{\partial L}{\partial(\theta e_n)} = 0, \quad n = 1, 2, \ldots, N \tag{56}$$

leads to the equations

$$C\prod_{m=1, m\neq n}^{N}(\theta e_m) - \frac{2\lambda\Delta A_n^2}{(\theta e_n)^3} = 0, \quad n = 1, 2, \ldots, N \tag{57}$$

We multiply (57) by θe_n, $n = 1, 2, \ldots, N$ and sum up all N equations to yield

$$NV - 2\lambda\left(\frac{\Delta A_1^2}{(\theta e_1)^2} + \frac{\Delta A_2^2}{(\theta e_2)^2} + \ldots + \frac{\Delta A_N^2}{(\theta e_N)^2}\right) = 0 \tag{58}$$

Bearing in mind Eqs. (54) and (58) becomes

$$\lambda = \frac{N}{2}V \tag{59}$$

Substituting Eq. (59) into Eq. (57) leads to

$$\frac{V}{(\theta e_n)} - N\frac{\Delta A_n^2}{(\theta e_n)^3}V = 0 \tag{60}$$

which implies that

$$\theta e_n = \sqrt{N}\Delta A_n, \quad n = 1, 2, \ldots, N \tag{61}$$

If $\sqrt{N}\Delta A_n$, $n = 1, 2, \ldots, N$ have the common factor, then let the ellipsoid radius θ be equal to the common factor, i.e.

$$\theta = \text{commonfactor}\{\sqrt{N}\Delta A_n,\ n = 1, 2, \ldots, N\} \tag{62}$$

otherwise, let the ellipsoidal radius θ be equal to unity, i.e.

$$\theta = 1 \tag{63}$$

Thus, the semi-axes of the ellipsoid are determined from the expressions

$$e_n = \frac{\sqrt{N}\Delta A_n}{\theta} \quad , n = 1, 2, \ldots, N \tag{64}$$

The one-to-one relationship of an interval vector and an ellipsoid is determined. Given one of an interval vector and an ellipsoid, we can determine another.

6.2. Comparison between interval analysis method and convex models

From Eqs. (34) and (42), we can compute the upper and lower bounds on the critical buckling load of composite structures by convex model method and interval analysis method, respectively. In this section, we will compare the accuracy of convex model method and interval analysis method in solving the buckling problem.

In terms of Eq. (52), from Eqs. (19) and (38), we can deduce the following equality:

$$v_c(\xi, \tau) = v_0(\xi, \tau) \tag{65}$$

For the second parts of Eq. (26) and (39), let us consider Chaucy–Schawrz inequality

$$\sum_{n=1}^{N} a_n b_n \leqslant \left(\sum_{n=1}^{N} a_n^2 \right)^{1/2} \left(\sum_{n=1}^{N} b_n^2 \right)^{1/2} \tag{66}$$

where a_i, b_i, $i = 1, 2, \ldots, N$ are real numbers.

Letting

$$a_n = |((1 + \psi_n(\tau)) \sin(n\pi\xi))\Delta A_n|, \quad b_n = \frac{1}{N} \tag{67}$$

from Eq. (66), we have

$$\sum_{n=1}^{N} |((1 + \psi_n(\tau)) \sin(n\pi\xi))|\Delta A_n \leqslant \sqrt{\sum_{n=1}^{N} N((1 + \psi_n(\tau)) \sin(n\pi\xi)\Delta A_n)^2}$$

$$= \sqrt{\sum_{n=1}^{N} ((1 + \psi_n(\tau)) \sin(n\pi\xi)\sqrt{N}\Delta A_n)^2} = \sqrt{\sum_{n=1}^{N} ((1 + \psi_n(\tau)) \sin(n\pi\xi)\theta e_n)^2} \tag{68}$$

Hence, because of Eqs. (65) and (68), from Eqs. (26) and (39), we obtain

$$v_{\min}(\xi, \tau) \leqslant \underline{v}(\xi, \tau) \leqslant \overline{v}(\xi, \tau) \leqslant v_{\max}(\xi, \tau) \tag{69}$$

from which, we can know that the upper and lower bounds on the critical buckling loads are calculated by the interval analysis method is sharper than those that are obtained by convex model method.

7. Numerical examples

Let us take the numerical example shown in Fig. 3 from Lindberg (1965) to show that comparison between models and interval analysis models in predicting the dynamic response and buckling measure set for structures with unknown-but-bounded initial imperfections.

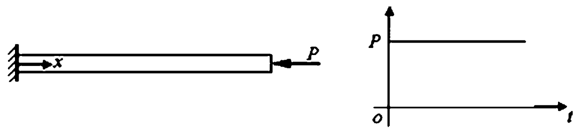

Fig. 3. A compressed bar and its applied axial load P.

In the case of determining the interval vector from an ellipsoid, the ellipsoidal set of allowed dominant initial imperfection Fourier coefficients is a sphere centered at the origin in 10-dimensional real number space. Thus, 10 harmonics are considered, the weighting matrix of the ellipsoid is taken as the 10×10 identity matrix and the radius of the ellipsoid is taken as $\theta = 0.07$. Thus, the interval vector of the initial imperfections of the bar can be determined as $A_n^{\mathrm{I}} = [0.045, 0.055]$, $i = 1, 2, \ldots, 10$ based on the given ellipsoid. The upper and lower curves of Fig. 4 show the maximum and minimum normalized total deflections as functions of normalized time, at the midpoint $\xi = 0.5$ of the bar. The axial load is twice the classical buckling load, so $\alpha = 2$. It can be seen from Fig. 4 that under the condition of the interval vector determined from an ellipsoid of the initial imperfection vector, the total deflection set obtained by convex models is smaller that by interval analysis method for the bar with initial imperfections. Namely the lower bound on the total deflection of the bar within interval analysis method is smaller than one predicted by convex models, and the upper bound on the total deflection of the bar furnished by the interval analysis method is larger than one yielded by convex models.

In the condition of determining the ellipsoid from an interval vector, the component of the interval vector of allowed dominant initial imperfection Fourier coefficients are $A_n^{\mathrm{I}} = [0.045, 0.055]$, $i = 1, 2, \ldots, 10$. Making use of Eqs. (29) and (30), the mean values and the radiuses of the interval vector are calculated $A_n^{\mathrm{c}} = 0.05$, $\Delta A_n = 0.005$, $i = 1, 2, \ldots, 10$. Thus, by Eqs. (52), (62) and (64), we can obtain the central values, the radius and semi-axes are, respectively, $A_{n0} = 0.05$, $i = 1, 2, \ldots, 10$; $\theta = 0.005\sqrt{10}$; $e_i = 1$, $i = 1, 2, \ldots, 10$. Fig. 5 portrays the dependence of the maximum and minimum total deflections of the bar with initial imperfections on the time by convex models and interval analysis method. From Fig. 5 we can see that the maximum and minimum total deflections of the bar non-linearly vary with the time τ; Convex models yield broader bounds, namely the lower bound within convex models is smaller than that predicted by interval models. Likewise, the upper bound furnished by the convex models is larger than that yielded by interval models.

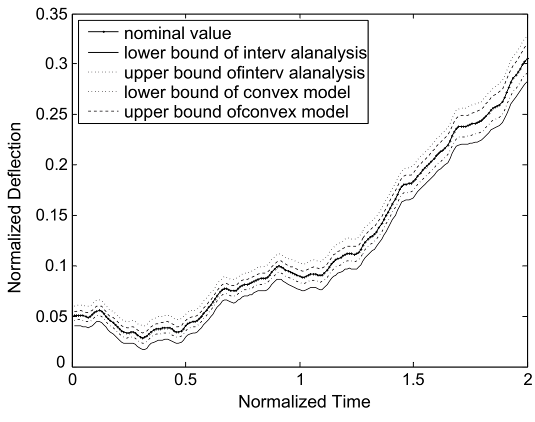

Fig. 4. The bounds on the total normalized deflection versus normalized time.

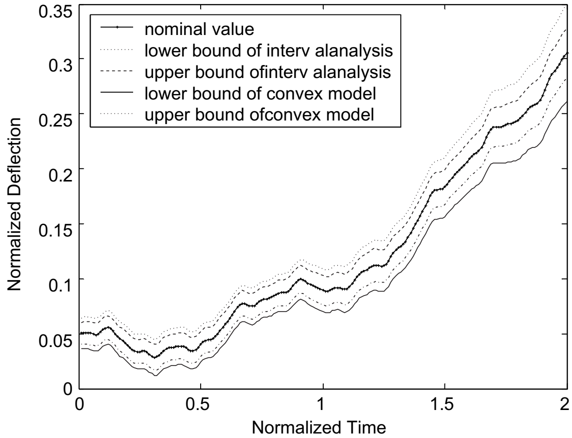

Fig. 5. The bounds on the total normalized deflection versus normalized time.

8. Conclusions

Two set estimation methods have been given to the problem of set estimating the dynamic response and buckling failure measures of bars, when the initial imperfection vector is uncertain except for the fact that it belongs to given bounded convex set. The cases of both ellipsoid constraints and interval constraints for the initial imperfections have been considered. In the former case, the interval vector containing the set of possible responses of the dynamic response and buckling failure measures of bars by convex models is given. In the latter case, the interval vector describing a bounding box or hyperrectangle as a set of possible dynamic responses is also derived. The transformation relationship of ellipsoid and interval vector was generally produced in different conditions. It is proved that, under the condition of the ellipsoid from an interval vector, the interval vector set yielded by interval analysis models is smaller than that presented by convex models; under the condition of the interval vector from an ellipsoid, the interval response vector set by convex models is smaller than that yielded by interval analysis models. Therefore, if we know the form of convex set of initial imperfections, we should use convex models for solving the dynamic response and buckling failure measures of bars with uncertain-but-bounded initial imperfections; if we describe the initial imperfections by an interval vector, we should employ interval analysis models for estimating the dynamic response and buckling failure measures of bars with uncertain-but-known initial imperfections. In fact, the interval vector containing the uncertain initial imperfections is more easily available than the ellipsoidal, so the interval analysis method is more practical.

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